Quantum Linear Prediction For System Identification And Spectral Estimation Applications

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***Abstract*—This paper presents the design and implementation of quantum algorithms and circuits for linear prediction. The intended applications are system identification, spectral estimation, and speech processing. A frequency-domain method that uses the quantum Fourier transform is developed to estimate the autocorrelation sequence of the signal and a quantum circuit is designed to estimate the linear prediction parameters. The quantum linear prediction performance is evaluated in system identification, autoregressive (AR) spectral estimation, and speech analysis applications. More specifically, we evaluate the performance of the quantum linear predictor in terms of quantum noise effects, qubit precision, and overall computational requirements. Comparisons of quantum versus classical linear prediction are presented.**

Keywords—Quantum linear prediction, quantum Fourier transform, HHL algorithm, qubits.

I. Introduction

Quantum computing (QC) promises to enhance computations and benefit several areas including big data, signal processing and machine learning [1]. Research efforts focus on exploring QC in various real-world applications [2]. QC has the potential for improved speed by leveraging properties of quantum mechanics such as *superposition* and *entanglement* [3].

In our brief literature review here, we cite work related to signal processing applications. In particular, the study presented here involves quantum computations related to matrix algebra, filtering and Fourier transforms. Previous research on relevant topics examined various capabilities of QC including quantum phase estimation for eigenvalue calculation [4], quantum filtering methods [5] and the utility of quantum Fourier transforms (QFT) [6] in Shor's algorithm to estimate the period of a function. Recent research has also examined how QFTs can be applied to signal analysis and synthesis [7]. Moreover, several studies have explored the use of QC in image and speech processing [8,9] as well as spectral estimation [10]. A survey paper on quantum information processing [11] also covered applications in machine learning and communication systems.

The study presented in this paper focuses on the design of quantum circuits for linear prediction (LP). LP has been heavily used for speech analysis-synthesis and compression. Applications in speech coding rely on the source-system model where the estimated LP parameters are used to produce an AR model that represents the frequency response [12-14] of the vocal tract. Atal provided an overview of the history of LP [15]. There are several variations of classical linear prediction (CLP) that appeared in the literature including lattice structures for LP, adaptive forward-backward prediction [16] and pole-zero prediction [17].

This paper proposes the design and implementation of quantum circuits for LP. The algorithm developed in this paper uses quantum circuit blocks for LP and the process is hereafter called quantum linear prediction (QLP). As is the case with CLP, QLP also requires autocorrelation sequence computation and then solution of the associated autocorrelation equations. For the autocorrelation computation, we design an algorithm that uses QFTs for fast convolution. For the solution of the linear autocorrelation equations. we use a modified version of the Harrow-Hassidim-Lloyd (HHL) algorithm [18]. The resulting QLP is evaluated in various applications and its performance is compared against the CLP.

The rest of the paper is organized as follows. Section II provides an overview of CLP. Section III and IV describes the basics of QC and the autocorrelation computation using the QFT circuit. Section V and VI explain the basic HHL algorithm used and the problems encountered with its application in speech analysis. Furthermore, a modified HHL to customize the algorithm for speech analysis is proposed. Section VII describes the details of the QLP algorithm. In Sections VIII and IX, we present QLP results in speech processing applications. Finally, in Section X, we present our concluding remarks.

II. Classical Linear Prediction

Linear prediction estimates the current sample of a sequence using a linear combination of past samples. The forward prediction error in CLP is written as

$$e(n) = x(n) - \hat{x}[n] = x(n) - \sum_{k=1}^{p} a_k . x[n-k] \qquad (1)$$

where, p is the order and a_k are LP coefficients, used to form an AR model. By minimizing the mean square prediction error, the autocorrelation equations are obtained.

$$r_i = \sum_{k=1}^{p} a_k . r_{|i-k|} \,, \quad 1 < i < p \qquad (2)$$

These equations have a Toeplitz symmetric structure which is shown in the matrix ($\boldsymbol{R}$) which is typically inverted using the Levinson-Durbin [12] algorithm.

$$\boldsymbol{R} = \begin{bmatrix} r_x(0) & r_x{}^*(1) & \dots & r_x{}^*(p) \\ r_x(1) & r_x(0) & \dots & r_x{}^*(p-1) \\ \vdots & \vdots & \ddots & \vdots \\ r_x(p) & r_x(p-1) & \dots & r_x(0) \end{bmatrix} \qquad (3)$$

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The autocorrelation equations in terms of the matrix $\boldsymbol{R}$ are given by

$$\boldsymbol{r} = \boldsymbol{R}\,\boldsymbol{a} \qquad (4)$$

where,

$$\boldsymbol{r} = [r_x(1)\, r_x(2) \ldots r_x(p)] \quad and\ \boldsymbol{a} = [a_1 a_2 \ldots a_p] \qquad (5)$$

As mentioned before, LP has several applications including system identification, speech analysis and spectral estimation.

A. System Modeling and Identification using LP

LP has been used in Autoregressive (AR) time series estimation problems. In terms of system identification, it can be used to model an unknown system from its output samples using an all-pole model. In our study, we will implement the autocorrelation method for LP [19] using quantum circuits.

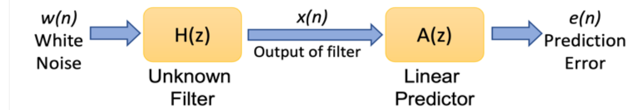


Fig 1. Block diagram of LP-based system identification.

B. Speech Analysis-Synthesis using LP

LP is most well known for its utility in speech analysis-synthesis. LP is used to model the speech spectrum by an AR model. This is done by minimizing the prediction error given in (1). Typically, in speech coding [12] the LP coefficients are estimated every 20ms assuming that the sampling rate is 8kHz. The LP-based speech analysis synthesis is given in Fig 2. A linear predictive coding (LPC) system quantizes and transmits the LP coefficients along with low bit rate representation of prediction error. Examples of algorithms using LP in adaptive differential pulse code modulation (ADPCM) and in the code excited linear predictor (CELP) are described in [12-14].

C. Spectral Estimation using LP

In parametric spectral estimation [20], the signal is represented by its AR time series and the coefficients of the AR model can be computed by minimizing the prediction error. The error $\mathrm{E}(z)$ in the z domain can be written as

$$E(z) = X(z)A(z) \qquad (6)$$

$$A(z) = 1 - \sum_{k=1}^{p} a_k . z^{-k} \qquad (7)$$

The polynomial $A(z)$ can be used to form the all-pole model $H(z) = 1/A(z)$. The power spectral density of the signal s(n) is then given by $S(e^{\omega}) = {\sigma_\omega}^2/|A(e^{\omega})|^2$ where ${\sigma_\omega}^2$ is the variance of white noise in the AR time series model. The spectral envelope that captures the overall spectral shape of speech signal is represented by $S(e^{\omega})$.

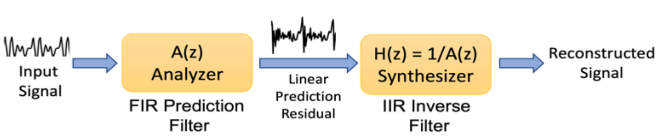


Fig 2. Analysis-synthesis of speech using linear prediction.

III. Quantum Computing Basics

Before we present the design of Quantum Linear Prediction circuits, we discuss briefly certain QC concepts. Qubits is the bit representation of the quantum system. Unlike classical bits, where we have two distinct states, a qubit can exist in a coherent superposition state of both. The principles of superposition, entanglement and interference provide the basis of QC. Entanglement specifically refers to interconnection of qubits, ensuring the correlation of qubit states, and interference describes the interaction of states among multiple qubits [3].

Furthermore, it is essential to study the effects of quantum noise on quantum systems. Quantum computers are susceptible to environmental interference, or quantum noise which effects the quantum measurement. We will explore the quantum noise effects on our developed quantum circuit designs.

For quantum encoding of a speech signal, we are using the Flexible Representation of Quantum Audio (FQRA) [8] process. As with nearly all classical signal processing systems, speech is processed using a sliding window which divides the signal into frames with 50% overlap to ensure smooth transitions at the signal frame boundaries. For every frame the autocorrelation equations are formed and the autocorrelation matrix is inverted to compute the LP coefficients.

IV. Autocorrelation Computation using the QFT

In LP, the autocorrelation sequence is first computed and the linear system of autocorrelation equations is formed as shown in equation (4). Calculating the autocorrelation coefficient estimation requires convolution of the signal with a time reversed version of itself, i.e. $x(n) * x(-n)$ where $*$ denotes convolution. This computation in the quantum domain presented several challenges due to inherent constraints of quantum mechanics [22]. In this study, we propose the computation of autocorrelations using QFTs. This was inspired by classical methods that use the FFT for fast convolution [23]. We use QFT and IQFT quantum circuit measurement results to obtain the quantum autocorrelation value.

A. The Quantum Fourier Transform (QFT)

Here, we present briefly the formulation of the QFT. Let a signal be encoded as a quantum state $|\psi\rangle$ with N states $|n\rangle$ and h_n amplitude, normalized by $norm = \sqrt{\sum_{n=0}^{N-1} {h_n}^2}$ where, $N = 2^m$ and m is the number of qubits. The QFT and IQFT for a basis state $|n\rangle$ can be calculated as

$$QFT|n\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i k n/N}\, |k\rangle \qquad (8)$$

$$IQFT|k\rangle \to \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{-2\pi i k n/N}\, |n\rangle \qquad (9)$$

A detailed derivation can be found in [6,7]. Fig 3 and 4 shows the constructed QFT and IQFT quantum circuits.

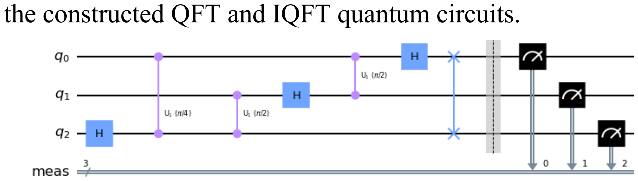


Fig 3. QFT circuit for 3 qubits after measurement.

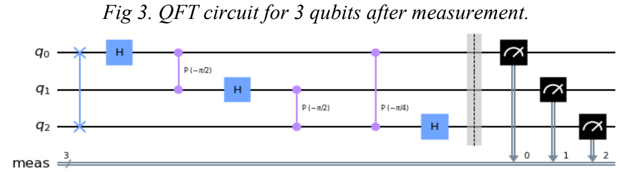


Fig 4. IQFT circuit for 3 qubits after measurement.

B. De-normalization of quantum results

The use of the QFT and IQFT for our study requires normalization and de-normalization. Because qubit measurement results in probabilistic estimation, direct comparison with classical DFT results require certain normalization steps. Therefore, de-normalization is necessary in order to compare quantum autocorrelation estimates with classical autocorrelation estimates. We develop the scaling factors as

$$QFT_f = norm * \sqrt{2^m}, \qquad IQFT_f = norm/\sqrt{2^m} \qquad (10)$$

These scaling factors enable us to evaluate quantum autocorrelation computations relative to classical results. This is explained further in the following sub-section.

C. Autocorrelation computation based on the QFT

We describe here the calculation of quantum autocorrelation sequence using QFTs as shown in Fig 5. We note that a circular effect can occur due to the periodic (circular) nature of discrete Fourier transform computations. To mitigate circular effects, the input speech signal is zero-padded to 2N length, and then normalized and encoded as a quantum state using the FRQA process. This quantum state is passed to the QFT circuit and the power spectrum (S_k) for the k-th frame of signal is obtained by

$$S_k \;=\; QFT_n\, QFT_n{}^* \qquad (11)$$

By encoding the power spectrum (S_k) as a quantum state and passing it to the IQFT circuit, the autocorrelation sequence $r_x(\tau)$ is obtained as

$$r_x(\tau) = IQFT\,|S_k\rangle \qquad (12)$$

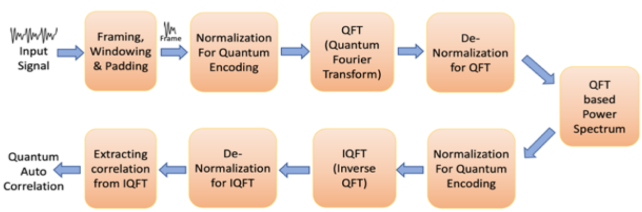


Fig 5: Block diagram of quantum autocorrelation calculation.

We compared the autocorrelation computation by calculating the Mean Squared Error (MSE) between the autocorrelation sequences r_x obtained using classical and quantum computing. We refer to the MSE in autocorrelation comparisons as MSE_r and we compute it for different number of qubits over several input frames. Table I displays averaged MSE_r results for 10 frames, revealing the increase of MSE_r with increase in the number of qubits. Low MSE_r values indicate that the QFT based quantum autocorrelation computation is consistent with the classical autocorrelation computation. However, the introduction of quantum noise models (for qiskit Aer simulator) significantly increased the MSE_r value.

TABLE I. MSE COMPARING CLASSICAL VS QUANTUM COMPUTED CORRELATIONS

Qubits	5	6	7	8
MSE_r	3.45e-32	6.65e-32	1.03e-31	1.33e-31
Quantum noise	1.76e-05	2.22e-05	3.37e-05	7.61e-05

V. THE HHL MATRIX INVERSION ALGORITHM

Since quantum LP requires inventing the autocorrelation matrix, we describe here an algorithm for matrix inversion that was developed for quantum systems, namely the HHL algorithm [18,19], which is the quantum algorithm that solves a linear system of equations and was shown to provide exponential speedup [18] for matrix inversion tasks relative to the classical methods [24]. The detailed HHL algorithm is shown in Fig 6. Let a general system of N linear equations be

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots A_{1N}x_N &= b_1 \\ A_{N1}x_1 + A_{N2}x_2 + \cdots A_{NN}x_N &= b_N \end{aligned} \qquad (13)$$

where, A_{ij} are the coefficients, b_i are the constant terms and x is the solution to the system. The matrix equation form is

$$Ax = b \qquad (14)$$

where $\boldsymbol{A}$ is an NxN matrix, $\boldsymbol{x}$ and $\boldsymbol{b}$ are column vectors. The paper [18] describes how the linear equation $\boldsymbol{Ax} = \boldsymbol{b}$ is solved in quantum domain. We will be describing this for the purpose of developing QLP; therefore, the $\boldsymbol{A}$ matrix corresponds to the **R** autocorrelation matrix of pxp size (p is the LP order), the $\boldsymbol{b}$ data vector and the $\boldsymbol{x}$ solution vector correspond to $\boldsymbol{r}$ autocorrelation sequence, and $\boldsymbol{a}$ coefficients respectively forming the matrix equation as $\boldsymbol{Ra} = \boldsymbol{r}$. The HHL algorithm requires that the matrix has the Hermitian structure. Since the autocorrelation is symmetric, $\boldsymbol{R}$ matrix can be inverted using the HHL. Representing vector $\boldsymbol{r}$ as $|r\rangle = \sum_{j=0}^{p-1} r_j\,|j\rangle$ normalized using $norm_r$ such that $\sum_{j=0}^{N-1} |r_j|^2 = 1$. After applying the Hamiltonian encoding e^{iRt} to $|r\rangle$, phase estimation technique is applied to calculate the eigenvalues λ_j and eigen vectors $|u_j\rangle$ of $\boldsymbol{R}$ matrix and to decompose $|r\rangle = \sum_{j=0}^{p-1} \beta_i\,|u_j\rangle$. The uncomputation is performed such that the state of the system becomes equal to

$$\sum_{j=0}^{p-1} \beta_i\, \lambda_j{}^{-1} |u_j\rangle = \boldsymbol{R}^{-1}|r\rangle = |a\rangle. \qquad (15)$$

This results in the quantum representation $|a\rangle$ of the solution vector $\boldsymbol{a}$ which is obtained after the measurement [18].

VI. PRE AND POST-PROCESSING OF SPEECH FOR HHL COMPUTATION

When operating on speech signals and determining the solution of linear autocorrelation equations, we encountered

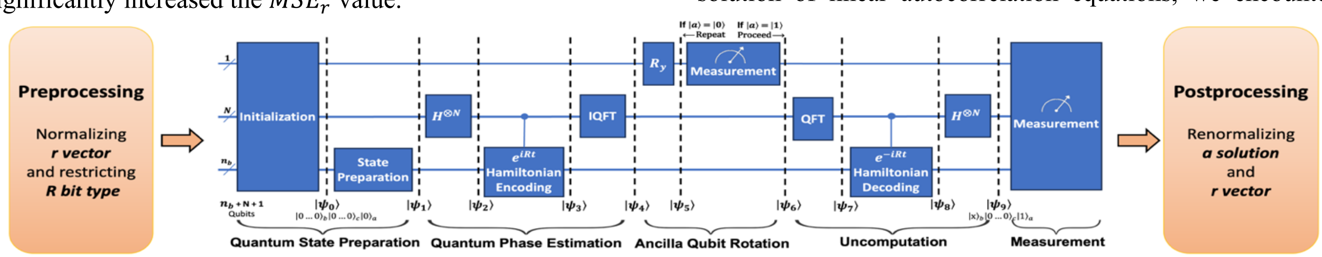


Fig 6: The modified HHL quantum circuit with pre and post processing blocks for speech signals.

several challenges. While the HHL algorithm can mathematically handle dense matrices, the solution becomes cumbersome. That is, the matrix becomes dense, the circuit depth grows exponentially with more complex entanglement patterns of qubits and hence, it becomes challenging to implement the algorithm. A comparison of sparse and dense matrix results is discussed in comparative results A section.

Although, a sparsity-independent quantum linear system algorithm based on quantum singular value estimation (QSVE) algorithm was proposed in [25], our unique approach aims to extend the applicability of the HHL algorithm specifically for signal processing. We develop a pre-processing structure for calculating the solution for the autocorrelation matrix of speech signals. Firstly, the 16-bit input speech signal, and **r** vector to be quantum encoded are normalized and converted to floating-point representation. Next, we restrict $\boldsymbol{R}\ matrix$ and $\boldsymbol{r}\ vector$ to be in float64 bit format with an attempt to reduce complex entanglement patterns such that computations are performed within memory-constrained requirements. However, this results in a loss of bit precision in representing numerical values, resulting in rounding errors carried through the HHL circuit. Since the measurement of quantum circuits results in probabilistic values, we developed a post-processing structure to produce a quantum solution that is close to the classical solution for the linear system of equations using (16).

$$LPC = |a\rangle_{measured} * \frac{Euclidean\ Norm}{Linear\ Norm} * norm_r \qquad (16)$$

Using the modified HHL algorithm with pre and post-processing stages (Fig 6), we are able to implement QLP for speech-processing applications.

VII. Quantum Linear Prediction

We develop a QLP algorithm using a QFT-based autocorrelation computation and the HHL algorithm for inverting the autocorrelation matrix. In the QLP process (Figure 7), the speech signal is divided into smaller segments, ranging from N = 32 to 128 samples, and windowed to reduce spectral leakage. Each frame is passed to the quantum speech encoding block. This is then quantum encoded for processing by the quantum autocorrelation computation system. The resulting autocorrelation sequence $\boldsymbol{r}$ and the pxp autocorrelation matrix $\boldsymbol{R}$ are then formed, pre-processed, and given as input to the quantum HHL circuit. The results after the measurement of the $|a\rangle_{measured}$ state have to be post-processed to produce the numerical LP coefficients. This is required to ensure that time series models and the vocal tract estimation is appropriately computed and consistent with classical computation.

Because of limitations in the HHL computation, QLP is designed for 4[th] order LP which captures the short-term quantum correlation. Low order LP is typically used in differential PCM for low bitrate communication with low computational complexity [26,27]. For CELP type coding [14,15], typically a 10[th] order predictor is used. Higher order predictors using quantum computing are currently being explored using the partitioned form of the matrix inversion lemma [28].

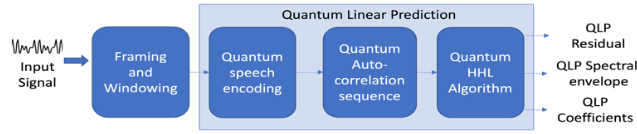


Fig 7: Quantum linear prediction block diagram.

VIII. Comparative Results

We analyzed the performance of the QLP and we compared it with CLP. We also documented the effect of qubit precision and quantum noise.

A. Results for sparse and dense linear equations

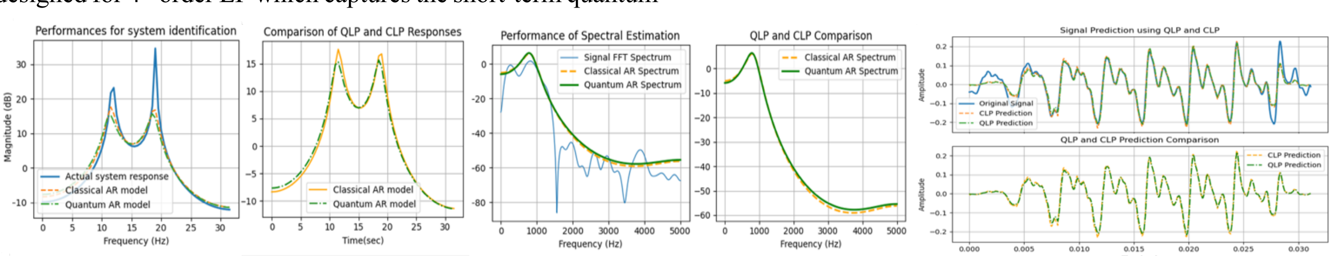
To examine the effects of matrix density and matrix size, we calculated the MSE_d between the classical and quantum solution results obtained for matrices with different densities (D) and matrix dimensions. The HHL computational complexity increases with increasing matrix density, therefore higher MSE_d values are observed. We analyzed the algorithm for different matrix dimensions and observed that MSE_d increases as the number of qubits increases. This occurs because as the matrix becomes more dense, the algorithm may loose accuracy [29]. The tradeoff between matrix density, dimension, and accuracy is shown in Table II.

TABLE II. MSE Comparison for Different Matrix Dimensions and Density

Matrix Dimension	Total Qubits	MSE_d for matrix with different density		
		D = 0.3	D = 0.6	D = 0.9
2x2	5	1.060e-29	1.4757e-7	0.000096
4x4	7	0.0000030	0.000013	0.000249
8x8	9	0.0000448	0.000061	0.004811

B. Effects of the number of qubits used for speech representation on the HHL accuracy

We also analyzed the effect of the number of qubits used for signal representation on the accuracy of the modified HHL algorithm. This was done by calculating the MSE between the classical and quantum LP solutions. This MSE is referenced as MSE_a. Table III shows that more qubits resulted in higher MSE_a, due to float64 datatype limitations and round-off errors. Though we have a higher MSE_a, we still need enough qubits to encode and represent the speech signals in the quantum state. Hence, it is essential to consider MSE when determining the appropriate number of qubits for signal encoding.



(a) System Identification using QLP and CLP (b) Short Time QLP & CLP Spectral Analysis (c) Speech Analysis Synthesis using QLP and CLP

Fig 8: Application results comparison of QLP and CLP.

TABLE III. MSE COMPARISON FOR DIFFERENT SPEECH SIGNAL QUBITS

Qubits	5	6	7	8
$\boldsymbol{MSE_a}$	0.0000182	0.0001611	0.000304	0.000876

IX. APPLICATION RESULTS

We investigated the effectiveness of QLP for system identification, spectral estimation, and speech analysis.

A. *Synthetic results of system identification*

For system identification, a 4^{th} order system with

$$H(z) = (1 - 0.24z^{-1} + 1.544z^{-2} - 0.21z^{-3} - 1.10z^{-4})^{-1} \quad (17)$$

is excited by white noise and used in our simulation. The QLP is applied on the output of this filter. The frequency response corresponding to $1/A(z)$ whose coefficients are computed with QLP is compared with the frequency response corresponding to $H(z)$. The two frequency response curves are shown in Fig 8 (a) and it can be seen they are close to one another.

B. *Results of AR spectral estimation*

Fig 8 (b) illustrates AR spectral estimation results for a given signal frame. As it can be seen, the quantum computed AR spectrum captures the shape of the input signal spectrum.

C. *Results of AR speech analysis synthesis*

For speech analysis-synthesis, as depicted in Fig 8 (c), CLP and QLP display similar speech prediction for a speech segment, providing high reconstruction accuracy.

X. CONCLUSION

In this paper, we developed and evaluated a quantum linear prediction system. The QLP algorithm is based on autocorrelations that are computed using the QFT. The autocorrelation matrix inversion is based on a modified HHL algorithm. Normalization was needed at several stages of the algorithm in order to ensure consistency with classical LP. The QLP was evaluated in several applications including speech analysis-synthesis and spectral estimation. Our study also considered the effect of the number of qubits used in signal representation on the overall quality of the estimates. Our simulation results for 4^{th} order QLP with real and synthetic signals showed consistency with classical prediction. One of the observations from these simulations is that there are important tradeoffs among the requires qubits for signal encoding, autocorrelation matrix density, and size to achieve the sufficient accuracy in the QLP solution. Future work, will examine expanding the order of QLP at least to 10 and also examining in detail the effects of quantum noise.

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